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Response of the Hopfield-Little model in an applied field

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An earlier study of hysteresis in driven neural networks [Phys. Rev. E **49**, R4811 (1994)] is extended to cover a larger range of storage ratios. Singularities in the response of the network are examined numerically, and compared with other models based on the zero-temperature Glauber dynamics of Ising spins in quenched random disorder. [S1063-651X(97)13407-9]

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I. INTRODUCTION

The purpose of this Brief Report is to update and extend an earlier study [1] of hysteresis in driven neural networks. In Ref. [1], we presented numerical results for a system of 200 spins, and a large storage ratio $\alpha=2$ (α denotes the ratio of the number of stored patterns to the number of spins in the system). These simulations showed pronounced hysteresis effects, and illustrated differences between the Hopfield-Little (HL) model of a neural network [2–5], the Sherrington-Kirkpatrick (SK) model of a spin glass [6], and the random field model [7]. Large remanence (approximately 20% of the saturation value) in the HL model with a storage ratio as high as $\alpha=2$ indicated that the retrieved pattern retains a finite overlap with the starting pattern even in the so-called spin-glass phase of the neural network. The magnetization curve in Ref. [1] showed large irregular jumps. This is in part due to the small size of the network which could be studied at that time. Figure 1 shows the data from 20 runs (i.e., different realizations of αN stored patterns) of a simulation of a larger ($N=2000$) system with $\alpha=2$. It indicates that the hysteresis loop may approach a smooth curve in the thermodynamic limit, but the remanence is there to stay.

Owing to the infinite-range interactions in the HL model, simulations of even small systems are very time consuming, more so with an applied field sweeping over a large range. It is clearly impractical to simulate systems over an exhaustive range of parameters. Only selected cases can be examined. In this paper, we limit ourselves to the hysteresis loops for relatively low and moderate values of the storage ratio α. For very small values of α, the hysteresis loop is necessarily rectangular, with the saturation magnetization reversing its direction at the coercive field. We ask the question, how does the rectangular loop go into a smooth loop (if it does) as α increases? Does the one discontinuity in the lower half of the hysteresis loop gradually move toward a smaller applied field and shrink in size (as happens in the random field model [7]), or does it break into several smaller discontinuities dispersed over a range of applied fields?

II. SIMULATIONS

The HL model of a neural network is based on the zero-temperature single-spin-flip dynamics of Ising spins. This is given by the following set of equations:

$$s_i=\mathrm{sgn}(\ell_i), \quad \ell_i=\sum_{j\neq i} J_{ij}s_j+h \quad (i,j=1,\ldots,N). \quad (1)$$

Here ℓ_i is the local field acting on the spin s_i, J_{ij} is the bond (pair interaction) between spins s_i and s_j, and h is a uniform applied field. In the HL model, the random bonds have the form

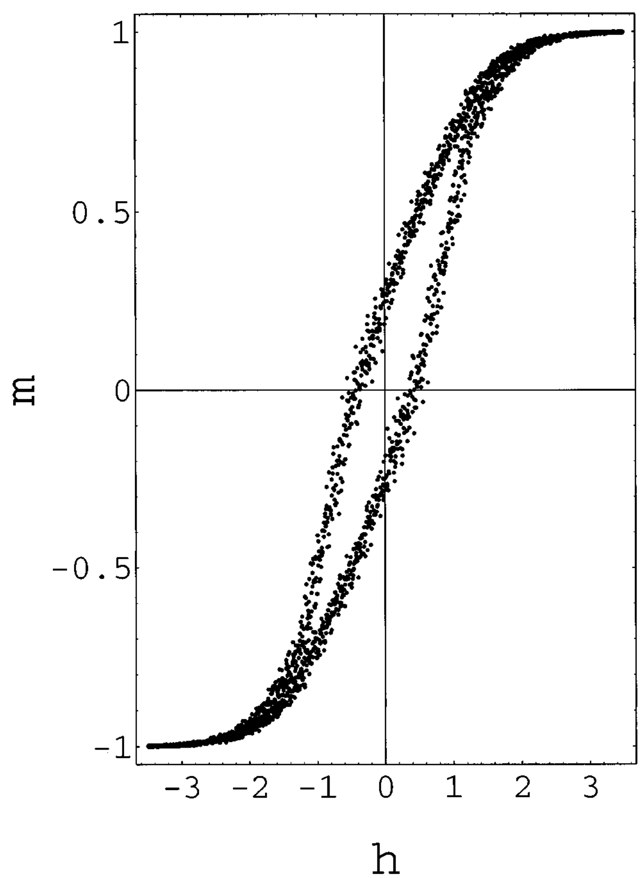

FIG. 1. Hysteresis in a neural network with $\alpha=2$ and $N=2000$. The figure shows data from 20 runs of a simulation. The dots represent the magnetization per spin (m) in an attractor of the dynamics in an applied field (h). The field was stepped up in intervals of $\Delta h=0.01$ to calculate the lower half of the hysteresis loop in an increasing field. The upper half of the loop was completed by symmetry, $m(h)=-m(-h)$. Although each single run shows irregular jumps in the magnetization curve (Barkhausen noise), the superposition of different runs suggests that the loop may become smooth in the thermodynamic limit.

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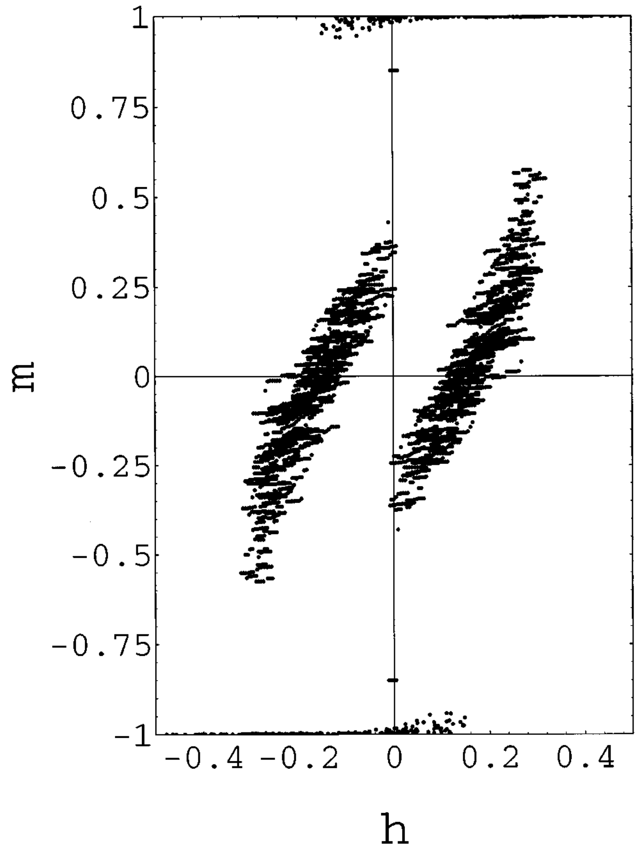

FIG. 2. Hysteresis in a neural network with $\alpha=0.15$ and $N=2000$. The dots represent the magnetization per spin (m) in applied field h shown on the x axis. Data from 100 runs of the simulation have been superimposed in the figure. There are two discontinuities in the magnetization in each half of the hysteresis loop.

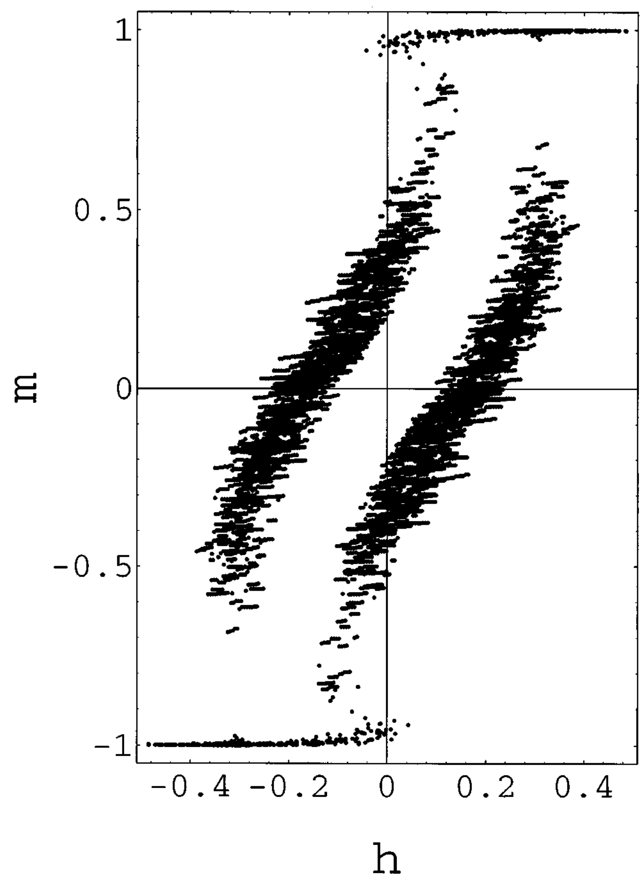

FIG. 3. Hysteresis in a neural network with $\alpha=0.20$ and $N=2000$ (100 runs). Note that the size of the vertical jumps in the magnetization has decreased as compared with Fig. 2, and the separation between the two jumps along the x axis has increased. These trends continue with increasing α leading to Fig. 1.

$$J_{ij}=\frac{1}{N}\sum_{\mu=1}^{\alpha N} \xi_i^\mu \xi_j^\mu , \qquad (2)$$

where ξ_i^μ takes the values 1 or -1 with equal probability. As in Ref. [1], we take one of the stored picture to have all spins up, $\{\xi_i^1=1; i=1,\ldots,N\}$. We could choose any other pattern as well, but the above choice is convenient because the conjugate field in this case is a uniform applied field. The remaining $\alpha N-1$ stored patterns are chosen randomly. The uniform applied field is swept from a large negative value to a large positive value in suitable steps, and the retrieval of its conjugate stored pattern is studied at each step. The results presented below are for a step size $\Delta h=0.005$. We also performed some simulations with step sizes $\Delta h=0.01$ and $\Delta h=0.1$. The results are essentially independent of the step size Δh. Spins are updated iteratively (in parallel) until an attractor of the dynamics is reached. The dynamics of the HL model always leads to a stable attractor; either a fixed point or a limit cycle of period 2 [8,1]. If the dynamics terminates in a fixed point, the magnetization in the fixed-point state gives the magnetization on the hysteresis loop. This fixed point then serves as the starting point for the next run of the dynamics in a higher applied field. If the dynamics terminates in a two-cycle limit, we calculate the energies of the two states which comprise the two-cycle limit, and choose the one which has the lower energy. This is used to obtain the magnetization, and to start the next run in a higher field. This process is continued until the applied field can support a fixed point with all spins up. The lower half of the hysteresis loop is obtained in the simulation. The upper half is completed by the relation $m(h)=-m(-h)$, which expresses an exact symmetry of the model. We studied systems of $N=500$, 1000, 2000, and 4000 spins for $\alpha=0.05$, 0.10, 0.15, and 0.20; making 100 runs in each case for $N=500$, 1000, and 2000, and ten runs for $N=4000$. We studied the largest jump in magnetization at each applied field in different runs of the simulation [9].

For $\alpha=0.05$, the magnetization jumps from -1 to $+1$ at an approximate applied field $h=0.3$. The value of the applied field at which the jump occurs is determined by the largest on-site field in the system resulting from the superposition of a given realization of $(\alpha N-1)$ random stored pictures. Because α is small, this quantity shows large fluctuations from sample to sample. However, the fluctuations decrease with increasing size of the system, as expected.

For $\alpha=0.10$ there is some indication that the magnetization jumps from -1 (approximately) to $+1$ (approximately) in two steps separated by a relatively smooth variation between the two jumps. However, the spacing (along the applied field axis) between the two jumps is small as compared with the statistical fluctuations in the position of the jumps in

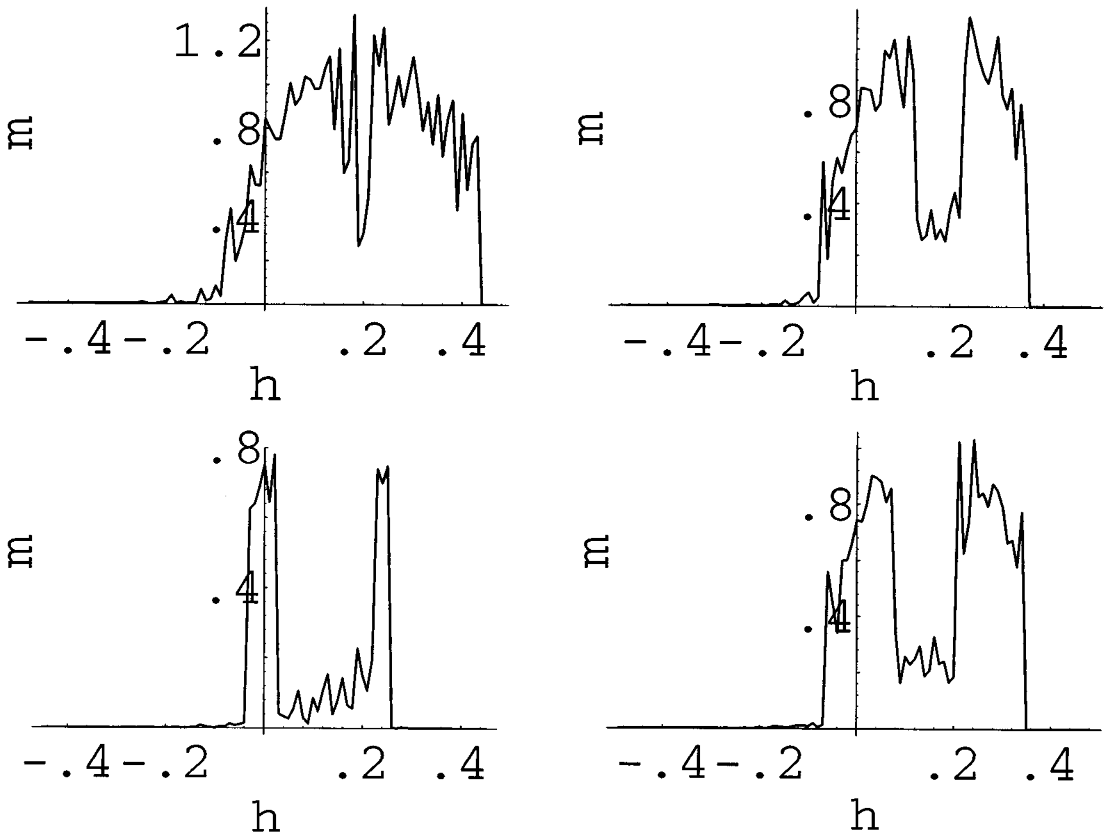

FIG. 4. Largest jumps in the magnetization for $\alpha=0.15$. The figures (clockwise, starting from top left) show the distribution of largest jumps in magnetization per spin in systems with $N=500$, 1000, 2000, and 4000 spins, respectively. The x axis shows the applied field, and the y axis the largest jump at that field in 100 runs of simulation (ten runs in case of $N=4000$). Peaks in the distribution which tend to sharpen with increasing system size indicate an instability in the thermodynamic limit.

different runs of the simulation. Thus the distribution of the largest jumps in the magnetization for $\alpha=0.10$ shows a single broad maximum at an applied field $h=0.1$ approximately.

For $\alpha=0.15$ and 0.20, there is a good indication of two jumps [10] in each half of the hysteresis loop, as seen in Figs. 2 and 3. A simple study of finite-size effects shows that the jumps in magnetization indicated in Figs. 2 and 3 become sharper with increasing size of the system. Figures 4 and 5 show the finite-size effects. These figures have only qualitative significance because the statistical fluctuations in the data are rather large. Their main significance is that there are two broad maxima in the figures which become better separated and sharper with increasing size of the system. This is indicative of two singularities in the system in the thermodynamic limit.

III. CONCLUDING REMARKS

The numerical simulations presented above suggest that the magnetization $m(h)$ of the HL model has two discontinuities as a function of the applied field h in a range of storage ratio extending on both sides of the critical storage $\alpha_c=0.14$. At smaller values of α ($\alpha\leqslant0.05$), there is only one noticeable discontinuity in the lower half of the hysteresis loop at an applied field ($h>0$). With increasing α, this discontinuity breaks up into two smaller discontinuities. The separation between the two discontinuities along the applied field axis increases with α. The first discontinuity in the magnetization in increasing field moves toward lower values of the applied field (it crosses the $h=0$ axis at approximately $\alpha=0.14$), and diminishes in size.

A simple and correct mean-field theory of the HL model as well as the SK model remains elusive (so far) in spite of a great effort over the past two decades. What this means is that the predictions of the extant mean-field theories do not match the numerical simulations perfectly. The reason is that the mean-field theories address an equilibrium state. Systems with strong quenched disorder are remarkably rich in metastable states, and the numerical simulations mostly scan the metastable states. The numerical simulations reveal interesting differences between random field and random bond models, as well as between different varieties of random bond models [11]. For example, there is a difference between the SK model of spinglass and the HL model for large α. Neural networks with $\alpha>0.14$ are sometimes referred to as being in the spin-glass state. However, the magnetization in the spin-glass phase is zero, but the overlap of the retrieved memory with the starting pattern does not drop to zero for $\alpha\geqslant\alpha_c=0.14$. It drops to 0.2 approximately, and then decreases rather slowly with increasing α. Mean-field theories fail to explain this effect [12,13]. The difference between the two models for large α probably comes from the small difference in the tails of the two distributions of bond strengths.

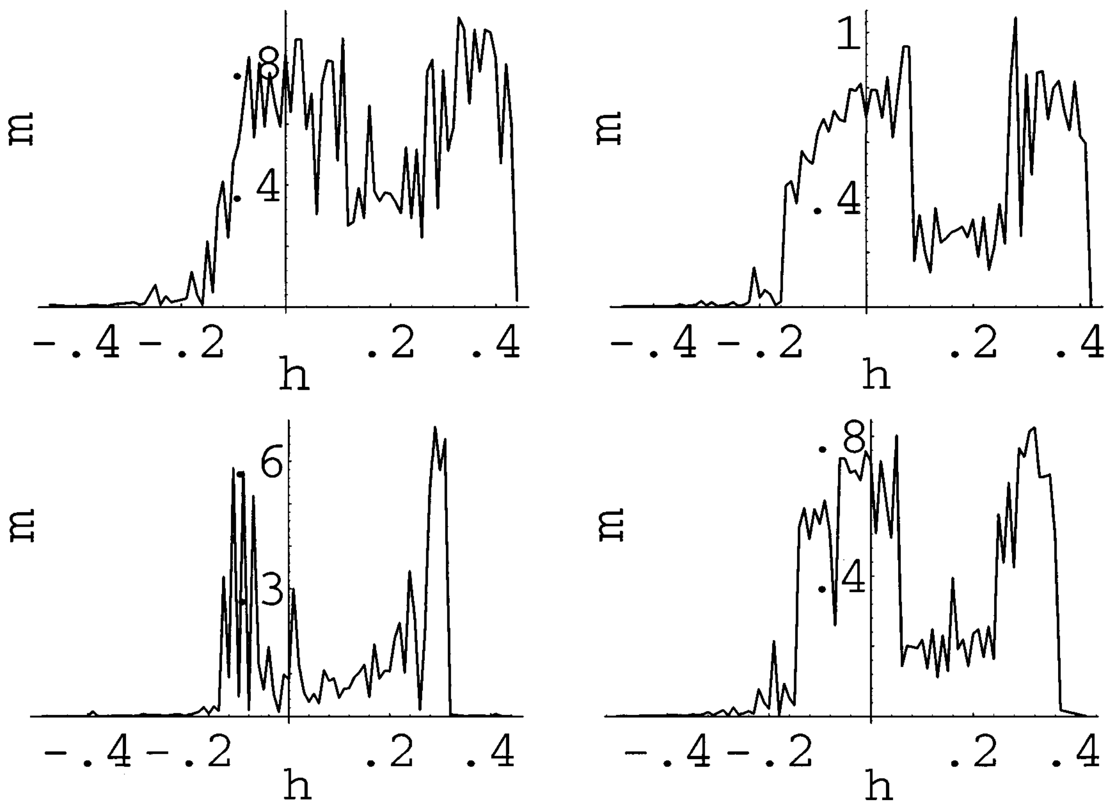

FIG. 5. Largest jumps in the magnetization for $\alpha = 0.20$. See the caption of Fig. 4 for other details.

The Gaussian distribution in the SK model allows for infinitely large deviations in the bond strengths even for finite N, but the distribution of the bond strengths in the HL model is bounded for finite N. An exact solution of simple one-dimensional cases [14] shows that the tails of the quenched distribution can have important bearings on the shape and singularities of the hysteresis loop. These singularities play an important role in determining the nature of the Barkhausen noise [15] associated with hysteresis loops [7].

In conclusion, numerical simulations of driven neural networks reveal a rich behavior which can be useful in understanding several aspects of hysteresis in magnets. It remains for us to understand the simulations in terms of a correct mean-field theory.

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